Perturbed WZW models and N=(2,2) supersymmetric sigma models with complex structure

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Abstract

We have perturbed N=(2,2) supersymmetric sigma models and WZW models on Lie groups by adding a term containing complex structure to their action. Then, by using non-coordinate basis we have shown that the conditions (from the algebraic point of view) for the preservation of the N=(2,2) supersymmetry account for the fact that the complex structure must be invariant. Also, we have shown that the perturbed WZW model with this term is an integrable model.

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1 Introduction

Supersymmetric sigma models are of particular interest, for their intimate connection to complex geometry of target manifold [1],[2] and for their role as effective low-energy actions for supergravity scalars. The N=(2,2) extended supersymmetry in sigma model from the geometrical point of view is equivalent to the existence of bi-Hermitian structure on the target manifold where the complex structures are covariantly constant with respect to torsionful affine connections [1] (see also [3] and references therein). We know that the algebraic structures related to bi-Hermitian relations of the N=(2,2) supersymmetric WZW models are the Manin triples [4],[5]. Furthermore, recently the algebraic structure associated to the bi-Hermitian geometry of the N=(2,2) supersymmetric sigma models on Lie groups has found in [6]. In [7], we have studied the perturbed N=(2,2) supersymmtric sigma models on Lie groups and have obtained conditions under which the N=(2,2) supersymmtry is preserved. Here in that direction, we perturb N=(2,2) supersummtric sigma models and WZW models on Lie groups by adding a term containing complex structure to the action. We show that for the preservation of the N=(2,2) supersymmtry we must have the invariant complex structure. Also, we show that the perturbed WZW model with similar term is an integrable model. The paper is organized as follows.

In section 2 we first review the N=(2,2) supersymmtric sigma models, then by using the method mentioned in [7] we show that the perturbed N=(2,2) supersymmtric sigma model on Lie group has N=(2,2) supersymmtry when the tensor structure in the perturbed term is an invariant complex structure. In section 3, we perturb WZW model with the bosonic version of the term which it used in section 2. Then, we show that this perturbed model is an integrable model when the tensor J is a complex structure. Finally, at the end of this section we present an example for the Heisenberg Lie group. Some concluding remarks are brought in section 4.

2 N=(2,2) supersymmetric sigma model on Lie groups perturbed with complex structure

We know that the N=(2,2) supersymmetric sigma models [1] on the manifold M can be written as a N=1 supersymmetric sigma models action as follows

$$S = \int d^2\sigma d^2\theta D_+\Phi^\mu D_-\Phi^\nu (G_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)), \tag{1}$$

where Φ^μ are N=1 superfields with bosonic parts as coordinates of the manifold M, and the bosonic parts of $G_{\mu\nu}$ and $B_{\mu\nu}$ are respectively the metric and the antisymmetric tensors on M. This action is manifestly invariant under supersymmetry transformations

$$\delta^1(\epsilon)\Phi^\mu = i(\epsilon^+ Q_+ + \epsilon^- Q_-)\Phi^\mu, \tag{2}$$

furthermore it has additional non manifest supersymmetry of the form

$$\delta^2(\epsilon)\Phi^\mu = \epsilon^+ D_+\Phi^\nu J^\mu_{+\nu}(\Phi) + \epsilon^- D_-\Phi^\nu J^\mu_{-\nu}(\Phi), \tag{3}$$

where in the above relations $Q_\pm$ and $D_\pm$ are supersymmetry generators and superderivatives respectively, and $J^\rho_{\pm\sigma} \in TM \otimes T^*M$ are complex structures. Invariance of the action (1) under the transformations (3) imposes the fact that $J_\pm$ must be bi-Hermitian complex structure such that their covariant derivations with respect to connections ($\Gamma^{\pm\mu}_{\rho\nu} = \Gamma^\mu_{\rho\nu} \pm G^{\mu\sigma}H_{\sigma\rho\nu}$) [1] are equal to zero [1]. In the case that M is a Lie group G, using non-coordinate basis, we have

$$G_{\mu\nu} = L_\mu{}^A L_\nu{}^B G_{AB} = R_\mu{}^A R_\nu{}^B G_{AB}, \tag{4}$$

$$H_{\mu\nu\lambda} = \frac{1}{2} L_\mu{}^A L_\nu{}^B L_\lambda{}^C H_{ABC} = \frac{1}{2} R_\mu{}^A R_\nu{}^B R_\lambda{}^C H_{ABC}, \tag{5}$$

$$J^\mu_{-\nu} = L^\mu{}_A J^A{}_B L_\nu{}^B, \qquad J^\mu_{+\nu} = R^\mu{}_A J^A{}_B R_\nu{}^B, \tag{6}$$

where G_{AB} is symmetric ad-invariant non-degenerate bilinear form (ad-invariant metric) and H_{ABC} is antisym-

[1] Note that H being the torsion three form $H_{\mu\rho\sigma} = \frac{1}{2}(B_{\mu\rho,\sigma} + B_{\rho\sigma,\mu} + B_{\sigma\mu,\rho})$

metric tensor on Lie algebra $\mathbf{g}$; furthermore $L_\mu{}^A(R_\mu{}^A)$ and $L^\mu{}_A(R^\mu{}_A)$ are left (right) invariant veilbien and their inverses on the Lie group G respectively and J is a Lie algebraic map; $J:\ \mathbf{g}\longrightarrow\mathbf{g}$. Then, the conditions of the N=(2,2) supersymmtric sigma model can be written in the following algebraic form [6]

$$J^2 = -I, \tag{7}$$

$$\chi_A + J^t\ \chi_A\ J^t + J^B{}_A\ \chi_B\ J^t - J^B{}_A\ J^t\ \chi_B = 0, \tag{8}$$

$$J^t\ G\ J = G, \tag{9}$$

$$H_A = J^t(H_B J^B{}_A) + J^t H_A J + (H_B J^B{}_A)J, \tag{10}$$

$$J^t(H_A + \chi_A G) = (J^t(H_A + \chi_A G))^t, \tag{11}$$

where $(\chi_A)_B{}^C = -f_{AB}{}^C$ is the adjoint representation such that $f_{AB}{}^C$ is the structure constant of the Lie algebra $\mathbf{g}$ and the matrix form $(H_A)_{BC} = H_{ABC}$, furthermore we have

$$(\chi_A G)^t = -\chi_A G. \tag{12}$$

These relations show that N=(2,2) supersymmetric sigma models on the Lie groups from the geometric point of view are corresponding to the bi-Hermitian structures on the Lie groups [1] or equivalently the algebraic bi-Hermitian structures (J, G, H) on the Lie algebras [6]. For N=2 supersymmetric WZW models on the Lie group G we have $H_{ABC} = f_{ABC}$. In this case, relations (8) and (9) show that we have the Lie bialgebra structures on $\mathbf{g}$ [5]; and relation (10) reduces to (8), and (11) is automatically satisfied i.e. Lie bialgebra structure is a special case of algebraic bi-Hermitian structure (J, G, H) with $H_{ABC} = f_{ABC}$ [6].

In [7], we have considered the general cases such that the perturbed N=(2,2) supersymmtric sigma models on Lie groups preserve N=(2,2) supersymmtry. Here we assume that the action (1) (as sigma models on Lie group or WZW model) has N=(2,2) supersymmetry, and as a especial example is perturbed with the following term

$$S^{'} = \int d^2\sigma d^2\theta D_+\Phi^\mu D_-\Phi^\nu G_{\mu\lambda}(\Phi) J^\lambda{}_\nu(\Phi), \tag{13}$$

where the bosonic part of a tensor $J^\lambda{}_\nu$ such that it is an element of $TM \otimes T^*M$. Now together with (1) we have

$$S'' = \int d^2\sigma d^2\theta D_+\Phi^\mu D_-\Phi^\nu (G''_{\mu\nu}(\Phi) + B''_{\mu\nu}(\Phi)), \tag{14}$$

such that

$$G''_{\mu\nu} = G_{\mu\nu} + \frac{1}{2}(G_{\mu\lambda}J^\lambda{}_\nu + G_{\nu\lambda}J^\lambda{}_\mu), \tag{15}$$

$$B''_{\mu\nu} = B_{\mu\nu} + \frac{1}{2}(G_{\mu\lambda}J^\lambda{}_\nu - G_{\nu\lambda}J^\lambda{}_\mu), \tag{16}$$

with the inverse metric

$$G''^{\mu\nu} = aG^{\mu\nu} + b(G^{\mu\lambda}J^\nu{}_\lambda + G^{\nu\lambda}J^\mu{}_\lambda). \tag{17}$$

Note that by using the condition $G''G''^{-1} = 1$ we have different values for a, b such that for obtaining the Hermitian condition (9) we must have $a = 1$. Note that in this case the coefficient of the b term is automatically zero. For this selection we have the **g)** case (final case) in [7] where the algebraic bi-Hermitian structure (J, G, H) is perturbed with $(0, 0, H')$. Now for having N=(2,2) supersymmtric sigma model, we must have the following relation (from relation (34) of [7])

$$H'_A J = (H'_A J)^t, \tag{18}$$

such that

$$H'_{ABC} = \frac{1}{2}(J^D{}_A f_{DCB} + J^D{}_B f_{DAC} - J^D{}_C f_{DAB}). \tag{19}$$

Now by substitution (19) in (18) and using (7)-(9) and (12) we obtain

$$J[X,Y] = [X, JY], \tag{20}$$

where this relation with condition $J^2 = -1$ shows that J is an invariant complex structure [8]. In this way, the N=(2,2) supersymmtry is preserved in the action (14) when the tensor $J^{\lambda}{}_{\nu}$ in (13) is an invariant complex structure [8]. Note that in [8] it is shown that any Lie algebra with invariant complex structure compatible with invariant metric admits an N=2 structure. Moreover, in that Ref. it is proved that these Lie algebras are the (even-dimensional) abelian Lie algebras. Therefore, in this way, we can perturbed the algebraic bi-Hermitian structure by the complex structure of any even-dimensional abelian Lie algebra such that the bi-Hermitian structure is preserved.

3 Perturbed WZW model with complex structure as an integrable model

We know that the WZW model based on Lie group G takes the following standard form [9].

$$\begin{aligned} S_{WZW}(g) \;&=\; \frac{k}{4\pi}\int_{\Sigma} d\xi^{+}\wedge d\xi^{-} \;< g^{-1}\partial_{+}g,\; g^{-1}\partial_{-}g > \\ &+\; \frac{k}{24\pi}\int_{B} < g^{-1}dg \;,\, [g^{-1}dg \;,\, g^{-1}dg] >, \end{aligned} \tag{21}$$

where the integrations are over worldsheet Σ and a 3-dimensional manifold B with boundary $\partial B = \Sigma$, respectively, and $g^{-1}dg$ with $g \in G$ is the left invariant one-form on Lie group G so that it may be expressed as

$$g^{-1}dg = (g^{-1}\partial_{\mu}g)^{A} X_A \partial_{\alpha}x^{\mu} d\xi^{\alpha} = L_{\mu}{}^{A} X_A \partial_{\alpha}x^{\mu} d\xi^{\alpha}, \tag{22}$$

where X_A is basis for the Lie algebra $\mathbf{g}$ of the Lie group G. The WZW action (21) then can be rewritten in the following form

$$S_{WZW}(g) = \frac{k}{4\pi}\int_{\Sigma} d^2\xi \; L_{\mu}{}^{A}\; G_{AB}\; L_{\nu}{}^{B} \partial_{+}x^{\mu}\partial_{-}x^{\nu} + \frac{k}{24\pi}\int_{B} d^3\xi \epsilon^{\alpha\beta\gamma} L_{\mu}{}^{A}\; G_{AD}\; L_{\nu}{}^{B} f^{D}{}_{BC}\; L_{\lambda}{}^{C} \partial_{\alpha}x^{\mu}\partial_{\beta}x^{\nu}\partial_{\gamma}x^{\lambda}, \tag{23}$$

where $G_{AB} = < X_A, X_B >$ is non-degenerate ad-invariant metric on $\mathbf{g}$. Now, we assume that the action (23) (as sigma model on Lie group) is perturbed with the following term

$$S^{'} = k^{'} \int d\xi^{+} d\xi^{-} (g^{-1}\partial_{+}g)^{A}\;\; G_{AD} J^{D}{}_{B}\;\; (g^{-1}\partial_{-}g)^{B}, \tag{24}$$

such that $J^{D}{}_{B}$ is an endomorphism of $\mathbf{g}$ i.e $J:\mathbf{g}\rightarrow\mathbf{g}$. Note that the indices $A, B, ...$ show the Lie algebra indices and Greek indices $\mu, \nu, ...$ show the Lie group manifold indices. Now, using the vielbein formalism we have

$$S^{'} = k^{'} \int d\xi^{+} d\xi^{-}\; L_{\mu}{}^{A}\;\; G_{AD} J^{D}{}_{B}\;\; L_{\nu}{}^{B}\, \partial_{+}x^{\mu}\partial_{-}x^{\nu} = k^{'} \int d\xi^{+} d\xi^{-}\; G_{\mu\lambda} J^{\lambda}{}_{\nu}\, \partial_{+}x^{\mu}\partial_{-}x^{\nu}. \tag{25}$$

Then the general action of the WZW model perturbed with the term (24) can be rewritten as follows

$$S^{''} \equiv\; S + S^{'} = \int d\xi^{+} d\xi^{-}\; L_{\mu}{}^{A}(G_{AB} + B_{AB} + k^{'}\, G_{AD} J^{D}{}_{B}) L_{\nu}{}^{B}\, \partial_{+}x^{\mu}\partial_{-}x^{\nu}, \tag{26}$$

such that $S^{''}$ as sigma model have invertible metric and anti-symmetric tensor as follows

$$\begin{aligned} G^{''}_{\mu\nu} &= L_{\mu}{}^{A} G_{AB} L_{\nu}{}^{B} \;+ \frac{k^{'}}{2}(L_{\mu}{}^{A} G_{AD} J^{D}{}_{B} L_{\nu}{}^{B} + L_{\nu}{}^{A} G_{AD} J^{D}{}_{B} L_{\mu}{}^{B}), \\ B^{''}_{\mu\nu} &= L_{\mu}{}^{A} B_{AB} L_{\nu}{}^{B} \;+ \frac{k^{'}}{2}(L_{\mu}{}^{A} G_{AD} J^{D}{}_{B} L_{\nu}{}^{B} - L_{\nu}{}^{A} G_{AD} J^{D}{}_{B} L_{\mu}{}^{B}), \end{aligned} \tag{27}$$

with the inverse metric

$$G''^{\mu\nu} = a\, L^{\mu}{}_{A} G^{AB} L^{\nu}{}_{B} \,+ b\,(L^{\mu}{}_{A} G^{AD} J^{B}{}_{D} L^{\nu}{}_{B} + L^{\nu}{}_{A} G^{AD} J^{B}{}_{D} L^{\mu}{}_{B}). \tag{28}$$

Similar to the previous section using $G'' G''^{-1} = 1$ and Hermitian condition $J^t G'' J = G''$ we will arrive at $a = 1$ such that the coefficient of the b term must be zero in $G''^{\mu\nu}$. Now we prove that the model (26) is an integrable model. For this propose we use the formalism presented in [10]. In this direction, we should calculate $H^{\lambda}{}_{\mu\nu}$, $\Gamma^{\lambda}{}_{\mu\nu}$ and $\Omega^{\lambda}{}_{\mu\nu}$; so after some calculation we have

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}(-f^{A}{}_{BC} + 2(\partial_{\rho} L_{\sigma}{}^{A}) L^{\sigma}{}_{B} L^{\rho}{}_{C}) L^{\lambda}{}_{A} L_{\mu}{}^{B} L_{\nu}{}^{C}, \tag{29}$$

$$H^{\lambda}{}_{\mu\nu} = \frac{1}{2}(h^{A}{}_{BC} + k'\, J^{D}{}_{B}\, f^{A}{}_{CD} - k'\, J^{A}{}_{D}\, f^{D}{}_{CB} - k'\, J^{D}{}_{C}\, f^{A}{}_{BD}) L^{\lambda}{}_{A} L_{\mu}{}^{B} L_{\nu}{}^{C}, \tag{30}$$

$$\begin{aligned} \Omega^{\lambda}{}_{\mu\nu} = \frac{1}{2}(&-h^{A}{}_{BC} - f^{A}{}_{BC} - k'\, J^{D}{}_{B}\, f^{A}{}_{CD} + k'\, J^{A}{}_{D}\, f^{D}{}_{CB} \\ &+ k'\, J^{D}{}_{C}\, f^{A}{}_{BD} + 2(\partial_{\rho} L_{\sigma}{}^{A}) L^{\sigma}{}_{B} L^{\rho}{}_{C}) L^{\lambda}{}_{A} L_{\mu}{}^{B} L_{\nu}{}^{C}, \end{aligned} \tag{31}$$

where in this calculation we use

$$h_{\mu\nu\lambda} = \frac{1}{2}(\partial_{\lambda} B_{\mu\nu} + \partial_{\nu} B_{\lambda\mu} + \partial_{\mu} B_{\nu\lambda}) = \frac{1}{2} L_{\mu}{}^{A} L_{\nu}{}^{B} L_{\lambda}{}^{C} h_{ABC}, \tag{32}$$

such that for the WZW model (by choosing $k = 4\pi$) we have

$$h_{ABC} = f_{ABC}. \tag{33}$$

In this way, we obtain a lax pair whose consistency conditions (a zero curvature representation) are equal to the equations of motion as follows[10]

$$\begin{aligned} &[\partial_{+} + \alpha_{\mu}(x) \partial_{+} x^{\mu}]\psi = 0, \\ &[\partial_{-} + \beta_{\nu}(x) \partial_{-} x^{\nu}]\psi = 0, \end{aligned} \tag{34}$$

where compatibility condition of the linear system yields the equation of motion, so that the matrices $\alpha_{\mu}(x)$ and $\beta_{\mu}(x)$ satisfy the following relations[10]

$$\beta_{\mu} - \alpha_{\mu} = \mu_{\mu}, \tag{35}$$

$$\partial_{\mu}\beta_{\nu} - \partial_{\nu}\alpha_{\mu} + [\alpha_{\mu}, \beta_{\nu}] = \Omega^{\lambda}{}_{\mu\nu}\mu_{\lambda}, \tag{36}$$

where equation (36) can then be rewritten as

$$F_{\mu\nu} = -(\nabla_{\mu}\mu_{\nu} - \Omega^{\lambda}{}_{\mu\nu}\mu_{\lambda}), \tag{37}$$

so that the field strength $F_{\mu\nu}$ and covariant derivative are written as follows

$$F_{\mu\nu} = \partial_{\mu}\alpha_{\nu} - \partial_{\nu}\alpha_{\mu} + [\alpha_{\mu}, \alpha_{\nu}], \qquad \nabla_{\mu} X = \partial_{\mu} X + [\alpha_{\mu}, X]. \tag{38}$$

Now for our model (26) we choose

$$\alpha_{\mu} = c\; C^{A}{}_{B} L_{\mu}{}^{B} X_{A}, \qquad \mu_{\mu} = d\; D^{A}{}_{B} L_{\mu}{}^{B} X_{A}, \tag{39}$$

where $\{X_A\}$ are the basis of the Lie algebra $\mathbf{g}$ and c,d are constants. By assuming that G_{AB} and $J^{A}{}_{B}$ are independent of the coordinate of the Lie group G; after some calculation we see that for satisfying relation (37) we must have the following relation

$$C^{A}{}_{B} = J^{A}{}_{B} + \delta^{A}{}_{B}, \qquad D^{A}{}_{B} = J^{A}{}_{B} + \delta^{A}{}_{B}, \tag{40}$$

with

$$c = \frac{k'}{k' + 2}, \qquad d = \frac{2}{k' + 2}\,, \tag{41}$$

such that $J^{A}{}_{B}$ must satisfy the (7)-(9), i.e., it must be a algebraic complex structure. In this way, we show that the WZW model (23) which is perturbed with (24) is integrable when J is a compatible complex structure on the Lie algebra $\mathbf{g}$. The equations of motion for this integrable model can be rewritten as the following forms

$$\begin{aligned} &[\partial_{+} + \frac{k'}{k' + 2}\,(J^{A}{}_{B} + \delta^{A}{}_{B}) L_{\mu}{}^{B} X_{A} \partial_{+} x^{\mu}]\psi = 0, \\ &[\partial_{-} +\; (J^{A}{}_{B} + \delta^{A}{}_{B}) L_{\nu}{}^{B} X_{A} \partial_{-} x^{\nu}]\psi = 0. \end{aligned} \tag{42}$$

3.1 An example

In the following we will give example for WZW model perturbed with complex structure on Heisenberg Lie group $\mathbf{A_{4,8}}$. Heisenberg Lie algebra with basis $\{X_1, .., X_4\}$ has the following set of non-trivial commutation relations [11]

$$[X_2, X_4] = X_2, \qquad [X_3, X_4] = -X_3, \qquad [X_2, X_3] = X_1. \tag{43}$$

Here, we obtain a non-degenerate ad-invariant metric by using the general solution of (12) as follows

$$G_{AB} = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & 0 & b \end{pmatrix}, \qquad a \in \Re - \{0\}, \quad b \in \Re. \tag{44}$$

In order to write (21) and (24) explicitly, we need $g^{-1}\partial_\alpha g$. To this end, we use the following parametrization of the Lie group G

$$g \;=\; e^{x^1 X_1} e^{x^2 X_2} e^{x^3 X_3} e^{x^4 X_4}, \tag{45}$$

where X_i and x^i are generators and coordinates of the Lie group, respectively. Inserting our specific choice of the parametrization (45) the $g^{-1}\partial_\alpha g$ take the following form [12]

$$g^{-1}\partial_\alpha g = (\partial_\alpha x^1)X_1 + (\partial_\alpha x^2)(x^3 X_1 + e^{x^4} X_2) + (\partial_\alpha x^3)(e^{-x^4} X_3) + (\partial_\alpha x^4)X_4, \tag{46}$$

from which we can read off the ${L_\mu}^A$'s and the terms that are being integrated over in(23) are calculated to be [2]

$${L_+}^A\, G_{AB}\, {L_-}^B = [\partial_+ x^1 \partial_- x^4 + \partial_+ x^4 \partial_- x^1 - \partial_+ x^2 \partial_- x^3 - \partial_+ x^3 \partial_- x^2 + x^3 \partial_+ x^2 \partial_- x^4 + x^3 \partial_+ x^4 \partial_- x^2], \tag{47}$$

$$\epsilon^{\alpha\beta\gamma} {L_\alpha}^A\, G_{AD}\, {L_\beta}^B (\mathcal{Y}^D)_{BC}\, {L_\gamma}^C = -2\epsilon^{\alpha\beta\gamma}\partial_\gamma[x^3 \partial_\alpha x^4 \partial_\beta x^2 - x^4 \partial_\alpha x^3 \partial_\beta x^2 - x^2 \partial_\alpha x^4 \partial_\beta x^3], \tag{48}$$

where in the above relations we choose $b = 0$, $a = -1$ in (44), and use the adjoint representation $(\mathcal{Y}^l)_{jk} = -{f_{jk}}^l$ and the following relation

$$L_\alpha \;\equiv\; g^{-1}\partial_\alpha g \;=\; (g^{-1}\partial_\alpha g)^A X_A = {L_\mu}^A X_A \partial_+ x^\mu. \tag{49}$$

On the other hand, using integrating by parts, the action (23) is reduced to

$$S_{WZW}(g) = \frac{k}{2\pi}\int_\Sigma d^2\xi \; (\partial_+ x^1 \partial_- x^4 - \partial_+ x^2 \partial_- x^3 + x^3 \partial_+ x^4 \partial_- x^2). \tag{50}$$

Now, for calculating of the perturbed term (24) we first choose complex structure compatible with metric (44) with $b = 0$, $a = -1$; from [6] we have

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{51}$$

by this selection, the action (24) takes the following form

$$S' = k' \int d^2\xi \; (e^{-x^4}\partial_+ x^1 \partial_- x^3 - e^{-x^4}\partial_+ x^3 \partial_- x^1 + x^3 e^{-x^4}\partial_+ x^2 \partial_- x^3 - x^3 e^{-x^4}\partial_+ x^3 \partial_- x^2). \tag{52}$$

Finally, perturbed WZW model (26) by choosing $k = 4\pi$ takes the following form

$$\begin{aligned} S = \int d^2\xi \; [k' e^{-x^4}\partial_+ x^1 \partial_- x^3 + 2\partial_+ x^1 \partial_- x^4 - (1 - k' x^3 e^{-x^4})\partial_+ x^2 \partial_- x^3 \\ - k' e^{-x^4}\partial_+ x^3 \partial_- x^1 - (1 + k' x^3 e^{-x^4})\partial_+ x^3 \partial_- x^2 + 2x^3 \partial_+ x^4 \partial_- x^2], \end{aligned} \tag{53}$$

[2]Note that we choose $\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in light cone coordinate.

such that the equations of motion can be rewritten in the following Lax form

$$[\partial_+ + \frac{k^{'}}{k^{'}+2}\{\ (\partial_+ x^1 + (x^3 - e^{x^4})\partial_+ x^2)X_1 + (\partial_+ x^1 + (x^3 + e^{x^4})\partial_+ x^2)X_2$$
$$+ (e^{-x^4}\partial_+ x^3 - \partial_+ x^4)X_3 + (e^{-x^4}\partial_+ x^3 + \partial_+ x^4)X_4\}]\psi = 0, \tag{54}$$

$$[\partial_- + \{\ (\partial_- x^1 + (x^3 - e^{x^4})\partial_- x^2)X_1 + (\partial_- x^1 + (x^3 + e^{x^4})\partial_- x^2)X_2$$
$$+ (e^{-x^4}\partial_- x^3 - \partial_- x^4)X_3 + (e^{-x^4}\partial_- x^3 + \partial_- x^4)X_4\}]\psi = 0. \tag{55}$$

4 Conclusion

We have proved that N=(2,2) sigma models when perturbed with complex structure on Lie groups can preserve N=(2,2) supersymmtry if their Lie algebras have invariant complex structure compatible with ad-invariant metric. Also, we have shown that the zero curvature representation and consistency of integrability condition for WZW models perturbed with this term are equivalent to the vanishing of the Nijenhuis tensor for the complex structure and existence of the compatible metric with this complex structure.

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